THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Tutorial 10
6th April 2022

1. (Ex.21 Ch.8 in textbook) We consider conformal mappings to triangles.

   (a) Show that
   $$\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}dz,$$
   with $0<\beta_1,\beta_2<1$, and $1<\beta_1+\beta_2<2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of 0,1 , and ∞, and with angles $\alpha_1\pi,\alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j+\beta_j=1$ and $\beta_1+\beta_2+\beta_3=2$.

   (b) What happens when $\beta_1+\beta_2=1$?

   (c) What happens when $\beta_1+\beta_2<1$?

   (d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is $\frac{\sin(\alpha_j\pi)}{\pi}\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\Gamma\left(\alpha_3\right)$.

Solution. (a) There is a small typo in our textbook, it should be $(1-z)^{-\beta_2}$ instead of $(1-z)^{\beta_2}$. Then we can use the Schwarz-Christoffel integral:
$$S(z)=\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}dz=e^{-\beta_2\pi}\int_0^z z^{-\beta_1}(z-1)^{-\beta_2}dz$$
Then by proposition 4.1, it maps $\mathbb{R}\cup\{\infty\}$ to a triangle $\mathbb{T}$ which mapping $0,\ 1,\ \infty$ to the vertexes. Because $\mathbb{H}$ is connected and $S(z)$ is a continuous map, $S(\mathbb{H})\subseteq\mathbb{C}\setminus\overline{T}$ or $S(\mathbb{H})\subseteq T$. Since holomorphic maps preserve the orientation, the only possible case is $S(\mathbb{H})\subseteq T$. Now we want to show it is conformal.

For injectivity, we use arguement principle, Let $\gamma(t):\mathbb{R}\to\mathbb{R}+i\epsilon$ be the curve very close to the real line. Then it is not difficult to see, $\forall w\in T$:
$$\int_{-\infty}^{\infty}\frac{(S(\gamma(t))-w)'}{S(\gamma(t))-w}dt=\int_{-\infty}^{\infty}\frac{(S(t)-w)'}{S(t)-w}dt=\log(S(t))|_{-\infty}^{\infty}=1\cdot 2\pi i$$
This tells us the zeros inside the triangle $\mathbb{T}$ of $S(z)-w$ is 1, i.e $S(z)$ is injective.

For surjectivity, if it is not surjective, then $Im(S(z))$ cannot be a simply connected region, but on the other hand $Im(S(z))$ is homeomorphic to $\mathbb{H}$ which is simply connected.

(b) The image will be the region bounded by two parallel lines and a line segment.

(c) The image will be the region bounded by two non-parallel lines and a line segment.

(d) Recall that the beta function:
$$B(\alpha,\beta):=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$
Thus
$$\int_0^1(1-t)^{-\beta_2}t^{-\beta_1}dt=\frac{\Gamma(1-\beta_2)\Gamma(1-\beta_1)}{\Gamma(2-\beta_1-\beta_2)}=\frac{\Gamma(\alpha_2)\Gamma(\alpha_1)}{\Gamma(1-\alpha_3)}=\frac{\sin\alpha_3\pi}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$$

$$\int_1^\infty (1-t)^{-\beta_2}t^{-\beta_1}dt = \int_1^0 (1-\frac{1}{s})^{-\beta_2}(\frac{1}{s})^{-\beta_1}\cdot\frac{-1}{s^2}ds = \int_0^1 (s-1)^{-\beta_2}s^{\beta_1+\beta_2-2}ds$$
$$= \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(1-\alpha_2)} = \frac{\sin\alpha_1\pi}{\pi}\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_1)$$

similarly applying $t = 1-\frac{1}{s}$ we get

$$\int_{-\infty}^0 (1-t)^{-\beta_2}t^{-\beta_1}dt = \frac{\sin\alpha_2\pi}{\pi}\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_1)$$

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2. (Ex.22 Ch.8 in textbook) If P is a simply connected region bounded by a polygon with vertices $a_1, ..., a_n$ and angles $\alpha_1\pi, ..., \alpha_n\pi$, and F is a conformal map of the disc $\mathbb{D}$ to P, then there exist complex numbers $B_1, ..., B_n$ on the unit circle, and constants c_1 and c_2 so that

$$F(z) = c_1\int_1^z \frac{d\zeta}{(\zeta-B_1)^{\beta_1}...(\zeta-B_n)^{\beta_n}} + c_2$$

Solution. Recall we have conformal map $G : \mathbb{D} \to \mathbb{H}$:

$$G(z) = i\frac{1-z}{1+z} \qquad G^{-1}(w) = \frac{i-w}{i+w}$$

then $\tilde{F} = F \circ G^{-1}$ is a conformal map from $\mathbb{H}$ to P. Thus it takes the following form:

$$\tilde{F}(w) = c_1\int_0^w \frac{d\zeta}{(\zeta-A_1)^{\beta_1}...(\zeta-A_n)^{\beta_n}} + c_2$$

Therefore

$$F(z) = c_1\int_1^z \frac{d(\frac{i-\zeta}{i+\zeta})}{(\frac{i-\zeta}{i+\zeta}-A_1)^{\beta_1}...(\frac{i-\zeta}{i+\zeta}-A_n)^{\beta_n}} + c_2$$
$$= c_1\int_1^z \frac{\frac{-2i}{(i+\zeta)^2}d\zeta}{(\frac{i-\zeta}{i+\zeta}-A_1)^{\beta_1}...(\frac{i-\zeta}{i+\zeta}-A_n)^{\beta_n}} + c_2$$
$$= c_1'\int_1^z \frac{d\zeta}{(\zeta-B_1)^{\beta_1}...(\zeta-B_n)^{\beta_n}} + c_2$$

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3. (Ex.24(a) Ch.8 in textbook) The elliptic integrals$K$ and K' defined for $0 < k < 1$ by

$$K(k) = \int_0^1 \frac{dx}{\left((1-x^2)\left(1-k^2x^2\right)\right)^{1/2}} \quad \text{and} \quad K'(k) = \int_0^{1/k} \frac{dx}{\left((x^2-1)\left(1-k^2x^2\right)\right)^{1/2}}$$

Show that if $\tilde{k}^2 = 1-k^2$ and $\tilde{k} > 0$, then

$$K'(k) = K(\tilde{k})$$

Solution. We follow the hint, by change of variable $x = \left(1 - \tilde{k}^2 y^2\right)^{-1/2}$, we have $dx = \tilde{k}^2 y \left(1 - \tilde{k}^2 y^2\right)^{-3/2} dy$, thus

$$
\begin{aligned}
K'(k) &= \int_0^{1/k} \frac{dx}{\left(\left(x^2-1\right)\left(1-k^2x^2\right)\right)^{1/2}} \\
&= \int_0^{1} \frac{\tilde{k}^2 y \left(1 - \tilde{k}^2 y^2\right)^{-3/2} dy}{\left(\left(\tilde{k}^2 y^2 \left(1 - \tilde{k}^2 y^2\right)^{-1}\right)\left(\tilde{k}^2 \left(1-y^2\right)\left(1 - \tilde{k}^2 y^2\right)^{-1}\right)\right)^{1/2}} \\
&= \int_0^{1} \frac{dy}{\left(\left(1-y^2\right)\left(1 - \tilde{k}^2 y^2\right)\right)^{-1/2}} \\
&= K(\tilde{k})
\end{aligned}
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